

THE GROWTH OF THE FIRST NON-EUCLIDEAN FILLING FUNCTION OF THE QUATERNIONIC HEISENBERG GROUP

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Abstract. The isoperimetric inequalities of the n -th quaternionic Heisenberg Group grow, up to dimension n , as fast as the ones of the Euclidean space. We identify the growth rate of the isoperimetric inequality in dimension $n+1$, which is strictly faster than the appropriate of the Euclidean space.

1 INTRODUCTION

Isoperimetric inequalities specify the maximal volume needed to fill a boundary of a given volume. A special class of such isoperimetric inequalities is formed by the filling functions. They describe the difficulty to fill Lipschitz cycles by Lipschitz chains. In [3] we proved Euclidean behaviour for the filling function in dimension m of the quaternionic Heisenberg Group $H_{\mathbb{H}}^n$ for $m \in \{2, \dots, n\}$, i.e. it grows like $l^{\frac{m+1}{m}}$. The known results for the filling functions of the complex Heisenberg Group $H_{\mathbb{C}}^n$ (see [7], [8]) suggest a growth like $l^{\frac{n+2}{n}}$ for the filling function of $H_{\mathbb{H}}^n$ in dimension $n+1$. Our technique in [3] could only confirm the super-Euclidean behaviour without telling the exact growth. Now we are able to state it:

Theorem 1. *Let $H_{\mathbb{H}}^n$ be the quaternionic Heisenberg Group of dimension $4n+3$, equipped with a left-invariant Riemannian metric.*

Then holds:

$$F_{H_{\mathbb{H}}^n}^{n+1}(l) \sim l^{\frac{n+2}{n}}.$$

This theorem was part of the author's dissertation [4] at the Karlsruhe Institute of Technology.

2 FILLING FUNCTIONS AND HEISENBERG GROUPS

2.1 FILLING FUNCTIONS

Filling functions describe the difficulty to fill a given boundary. Formally this is described in terms of *Lipschitz chains*. In the following let X be a metric space and $m \in \mathbb{N}$. Further we denote by \mathcal{H}^m the m -dimensional Hausdorff-measure of X and by Δ^m we denote the m -simplex equipped with an Euclidean metric.

Definition. A Lipschitz m -chain a in X is a (finite) formal sum $a = \sum_j z_j \alpha_j$ of Lipschitz maps $\alpha_j : \Delta^m \rightarrow X$ with coefficients $z_j \in \mathbb{Z}$.

The boundary of a Lipschitz m -chain $a = \sum_j z_j \alpha_j$ is defined as the Lipschitz $(m-1)$ -chain

$$\partial a = \sum_j \left(z_j \sum_{i=0}^m (-1)^i \alpha_{j| \Delta_i^m} \right)$$

where Δ_i^m denotes the i^{th} face of Δ^m .

A Lipschitz m -chain a with zero-boundary, i.e. $\partial a = 0$, is called a Lipschitz m -cycle.

A filling of a Lipschitz m -cycle a is a Lipschitz $(m+1)$ -chain b with boundary $\partial b = a$.

We define the mass of a Lipschitz m -chain a as the total volume of its summands:

$$\text{mass}(a) := \sum_j z_j \text{vol}_m(\alpha_j(\Delta^m)) .$$

In the case, that X is a Riemannian manifold, the volume of such a summand is given by $\text{vol}_m(\alpha_j(\Delta^m)) = \int_{\Delta^m} J_{\alpha_j} d\lambda$, where $d\lambda$ denotes the m -dimensional Lebesgue-measure and J_{α_j} is the jacobian of α_j . This is well defined, as Lipschitz maps are, by Rademacher's Theorem, almost everywhere differentiable.

Given a m -cycle, one is interested in the filling with the smallest mass. Then one varies the cycle and examines how large the ratio of the mass of the optimal filling and the mass of the cycle can get. This leads to an invariant of the space X , the *filling functions*:

Definition. Let $n \in \mathbb{N}$ and let X be a n -connected metric space. For $m \leq n$ the $(m+1)^{\text{th}}$ -filling function of X is given by

$$F_X^{m+1}(l) = \sup_a \inf_b \text{mass}(b) \quad \forall l \in \mathbb{R}^+,$$

where the infimum is taken over all $(m+1)$ -chains b with $\partial b = a$ and the supremum is taken over all m -cycles a with $\text{mass}(a) \leq l$.

As we are mostly interested in the large scale geometry of the space X , the exact description of the filling functions is of less importance to us. Indeed we only look at the asymptotic behaviour of the functions. We do this by the following equivalence relation, which makes the growth rate of the filling functions an quasi-isometry invariant.

Definition. Let $f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be functions. Then we write $f \preceq g$ if there is a constant $C > 0$ with

$$f(l) \leq Cg(Cl) + Cl + C \quad \forall l \in \mathbb{R}^+.$$

If $f \preceq g$ and $g \preceq f$ we write $f \sim g$. This defines an equivalence relation.

We read this notation $f \preceq g$ as “ f is bounded from above by g ” respectively “ g is bounded from below by f ” according whether we are more interested in f or g .

Proposition 2.1 (see for example [6, Lemma 1]). Let X and Y be n -connected metric spaces. Then holds:

$$X \text{ quasi-isometric to } Y \Rightarrow F_X^{j+1} \sim F_Y^{j+1} \quad \forall j \leq n.$$

Any two left-invariant Riemannian metrics on a Lie group are Lipschitz-equivalent. So one gets by the above proposition, that the behaviour of the filling function of a Lie group equipped with a left-invariant Riemannian metric does not depend on the choice of this metric.

Let's look at the example of the filling functions of the n -dimensional Euclidean space. They were first computed by Herbert Federer and Wendell H. Fleming in [2].

Example. *The filling functions of the Euclidean space \mathbb{R}^n are*

$$F_{\mathbb{R}}^{j+1}(l) \sim l^{\frac{j+1}{j}} \quad \text{for } j \leq n-1.$$

This enables us to use the terms *Euclidean*, *sub-Euclidean* respectively *super-Euclidean filling function* for filling functions with the same, strictly slower respectively strictly faster growth rate than $l^{\frac{j+1}{j}}$.

The following theorem generalises the Euclidean case to spaces with non-positive curvature. For a proof see [5].

Theorem 2.2. *The filling functions of an n -dimensional Hadamard space X are*

$$F_X^{j+1}(l) \preccurlyeq l^{\frac{j+1}{j}} \quad \text{for } j \leq n-1.$$

The fact that a Riemannian manifold with non-positive curvature has Euclidean or sub-Euclidean filling functions in all dimensions yields a sufficient criterion for positive curvature: Let M be a Riemannian manifold with a super-Euclidean filling function in some dimension, then there is some positive curvature on M .

2.2 HEISENBERG GROUPS

The *complex Heisenberg Group* $H_{\mathbb{C}}^n$ is a higher dimensional analogue of the classical 3-dimensional Heisenberg Group

$$\mathbf{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \leq \mathrm{GL}_3(\mathbb{R}).$$

Equipped with a left invariant Riemannian metric it appears as horosphere in the complex-hyperbolic space $\mathrm{SU}(n, 1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$. As abstract Lie group it can be defined as follows:

Definition. *The complex Heisenberg Group $H_{\mathbb{C}}^n$ of dimension $2n + 1$ is as manifold*

$$H_{\mathbb{C}}^n := \mathbb{C}^n \times \mathrm{Im} \, \mathbb{C}$$

where \mathbb{C} denotes the complex numbers. The group law is given by

$$(z, x)(w, y) := (z + w, x + y - \frac{1}{2} \sum_{i=1}^n \mathrm{Im}(z_i \overline{w_i})) .$$

This is a 2-step nilpotent Lie group with (real) Lie algebra $\mathfrak{h}_{\mathbb{C}}^n = V_1 \oplus V_2$ where $V_1 = \mathbb{C}^n$, $V_2 = \mathrm{Im} \, \mathbb{C} \cong \mathbb{R}$

and with the bracket

$$[(Z, X), (W, Y)] = (0, \sum_{i=1}^n \text{Im}(Z_i \overline{W_i})) .$$

It can be seen as the unique simply connected Lie group with Lie algebra generated by

$$B := \{j_1, \dots, j_n, k_1, \dots, k_n, K\}$$

and with the only non-trivial brackets of the generators

$$[k, j] = K \quad \text{if both elements in the bracket have the same index.}$$

The group we are examining in this paper is the *quaternionic Heisenberg Group* $H_{\mathbb{H}}^n$. It appears similar to the complex Heisenberg Group as horosphere, now in the quaternionic-hyperbolic space $\text{Sp}(n, 1)/(\text{Sp}(n) \times \text{Sp}(1))$.

Definition. The quaternionic Heisenberg Group $H_{\mathbb{H}}^n$ of dimension $4n + 3$ is as manifold

$$H_{\mathbb{H}}^n := \mathbb{H}^n \times \text{Im } \mathbb{H}$$

where \mathbb{H} denotes the Hamilton quaternions. The group law is given by

$$(z, x)(w, y) := (z + w, x + y - \frac{1}{2} \sum_{i=1}^n \text{Im}(z_i \overline{w_i})) .$$

This is a 2-step nilpotent Lie group with (real) Lie algebra $\mathfrak{h}_{\mathbb{H}}^n = V_1 \oplus V_2$ where $V_1 = \mathbb{H}^n$, $V_2 = \text{Im } \mathbb{H}$ and with the bracket

$$[(Z, X), (W, Y)] = (0, \sum_{i=1}^n \text{Im}(Z_i \overline{W_i})) .$$

It can be seen as the unique simply connected Lie group with Lie algebra generated by

$$B := \{h_1, \dots, h_n, i_1, \dots, i_n, j_1, \dots, j_n, k_1, \dots, k_n, I, J, K\}$$

and with the only non-trivial brackets of the generators

$$[a, h] = A \text{ for } a \in \{i_1, \dots, i_n, j_1, \dots, j_n, k_1, \dots, k_n\}, \text{ (here } A \text{ denotes the capital letter of the choice of } a\text{)}$$

and

$$[k, j] = I, \quad [i, k] = J, \quad [j, i] = K$$

if both elements in the bracket have the same index.

2.3 BURILLO'S FILLING THEOREM

For our proof of the lower bound on the $(n + 1)^{\text{th}}$ filling function of the quaternionic Heisenberg Group $H_{\mathbb{H}}^n$, we will apply a technique established by Burillo in [1].

Theorem 2.3 (see [1, Proposition 1.2]). *Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric and let $m \in \mathbb{N}$. If there exists a Lipschitz $(m + 1)$ -chain b and a closed G -invariant $(m + 1)$ -form γ in G and constants $C, r, s > 0$ such that*

$$1) \text{ mass}(s_t(\partial b)) \leq Ct^r ,$$

$$2) \int_b \gamma > 0 ,$$

3) $s_t^* \gamma = t^s \gamma$,

then holds $F_G^{m+1}(l) \succ l^{\frac{s}{r}}$.

Burillo used this criterion to compute lower bounds on the filling functions of the complex Heisenberg Group $H_{\mathbb{C}}^n$. We will use this criterion as well as the property that $H_{\mathbb{H}}^n$ contains $H_{\mathbb{C}}^n$.

3 THE PROOF OF THEOREM 1

In this section we compute the $(n+1)$ -dimensional filling function of the quaternionic Heisenberg Group $H_{\mathbb{H}}^n$. As $H_{\mathbb{H}}^n$ fulfils the conditions of [3, Theorem 1], we already have the super-Euclidean upper bound:

$$F_{H_{\mathbb{H}}^n}^{n+1}(l) \preccurlyeq l^{\frac{n+2}{n}}.$$

Thus we only have to prove the corresponding lower bound.

Proposition 3.1. *Let $H_{\mathbb{H}}^n$ be the quaternionic Heisenberg Group of dimension $4n+3$. Then holds:*

$$F_{H_{\mathbb{H}}^n}^{n+1}(l) \succcurlyeq l^{\frac{n+2}{n}}.$$

Proof. We will use Burillo's filling theorem (see Theorem 2.3). To this end, we have to construct a Lipschitz $(n+1)$ -chain b in $H_{\mathbb{H}}^n$ and a closed $H_{\mathbb{H}}^n$ -invariant $(n+1)$ -form η on $H_{\mathbb{H}}^n$ with the correct scaling behaviour.

For this we avail ourselves of the constructions Burillo did in the proof of [1, Theorem 2.1] to obtain the lower bound for the filling function $F_{H_{\mathbb{C}}^n}^{n+1}$ of the complex Heisenberg Group. We denote the there constructed $(n+1)$ -chain by b' and the corresponding $(n+1)$ -form by γ .

Now let $x_1, \dots, x_n, y_1, \dots, y_n, Z$ be the usual basis of the Lie algebra $\mathfrak{h}_{\mathbb{C}}^n$ of the complex Heisenberg Group $H_{\mathbb{C}}^n$ and h_1, \dots, k_n, I, J, K the basis of the Lie algebra $\mathfrak{h}_{\mathbb{H}}^n$ of the quaternionic Heisenberg Group $H_{\mathbb{H}}^n$ (compare Section 2.2).

Then the complex Heisenberg Group $H_{\mathbb{C}}^n$ embeds as Lie subgroup into the quaternionic Heisenberg Group $H_{\mathbb{H}}^n$. We do this on the Lie algebra level via the map

$$\Phi : \mathfrak{h}_{\mathbb{C}}^n \rightarrow \mathfrak{h}_{\mathbb{H}}^n \quad \text{defined by} \quad j_i \mapsto j_i, \quad k_i \mapsto k_i, \quad K \mapsto K.$$

Hence we can consider the in [1] constructed $(n+1)$ -chain b' as an $(n+1)$ -chain in the quaternionic Heisenberg Group $H_{\mathbb{H}}^n$. We set $b := b'$. The above embedding respects the grading of the Lie algebras, i.e. vectors of the first layer are mapped to vectors of the first layer and vectors of the second layer are mapped to vectors of the second layer. Therefore the boundary ∂b of the chain b has the same scaling behaviour in the quaternionic Heisenberg Group $H_{\mathbb{H}}^n$ as before in the the complex Heisenberg Group $H_{\mathbb{C}}^n$, i.e. $\text{mass}(s_t(\partial b)) \leq \text{mass}(\partial b) \cdot t^n$.

Consequently it only remains to construct a closed, $H_{\mathbb{H}}^n$ -invariant $(n+1)$ -form η on $H_{\mathbb{H}}^n$, such that η restricts on the embedded $H_{\mathbb{C}}^n$ to the in [1] constructed closed, $H_{\mathbb{C}}^n$ -invariant $(n+1)$ -form γ on $H_{\mathbb{C}}^n$ (this would imply condition 2) in Theorem 2.3) and satisfies

$$s_t^* \eta = t^{n+2} \eta.$$

The form γ is given (with respect to the above notation for $H_{\mathbb{C}}^n \subset H_{\mathbb{H}}^n$) by

$$\gamma = (-1)^n \cdot K^* \wedge h_1^* \wedge \dots \wedge h_n^*$$

where for $v \in \mathfrak{h}_{\mathbb{H}}^n$ the symbol v^* denotes the dual form of v .

We start the construction of η by defining some special n -forms. Let \mathfrak{S}_n be the set of all ($H_{\mathbb{H}}^n$ -invariant) n -forms S of the shape

$$S = h_l^* \wedge \dots \wedge h_m^* \wedge i_p^* \wedge \dots \wedge i_q^*$$

with increasing index within the h^* -part, increasing index within the i^* -part and such that every number in $\{1, \dots, n\}$ appears as index of an h^* or an i^* , as well with an even number of i^* 's. In particular, each integer between 1 and n appears exactly once as index. For example in the case $n = 3$ we get:

$$\mathfrak{S}_3 = \{h_1^* \wedge h_2^* \wedge h_3^*, h_1^* \wedge i_2^* \wedge i_3^*, h_2^* \wedge i_1^* \wedge i_3^*, h_3^* \wedge i_1^* \wedge i_2^*\}.$$

Further let \mathfrak{T}_n be the set of all ($H_{\mathbb{H}}^n$ -invariant) n -forms T of the shape

$$T = h_l^* \wedge \dots \wedge h_m^* \wedge i_p^* \wedge \dots \wedge i_q^*$$

with increasing index within the h^* -part, increasing index within the i^* -part and such that every number in $\{1, \dots, n\}$ appears as index of an h^* or an i^* , as well with an odd number of i^* 's. In particular, each integer between 1 and n appears exactly once as index. For example in the case $n = 3$ we get:

$$\mathfrak{T}_3 = \{i_1^* \wedge i_2^* \wedge i_3^*, h_1^* \wedge h_2^* \wedge i_3^*, h_1^* \wedge h_3^* \wedge i_2^*, h_2^* \wedge h_3^* \wedge i_1^*\}.$$

Each of this n -forms in \mathfrak{S}_n and \mathfrak{T}_n can be obtained either from h_1^* or from i_1^* by adding successively h_{r+1}^* or i_{r+1}^* in the r^{th} step at the last position of the h^* -part respectively at the last position of the i^* -part. If one now gives h_1^* the sign “+” and i_1^* the sign “−” this induces a sign to each of the n -forms $S \in \mathfrak{S}_n$ and $T \in \mathfrak{T}_n$ by the following rule:

Let A_r be the signed r -form before adding h_{r+1}^* or i_{r+1}^* . Then:

- If A_r has an even number of i^* 's, we change the sign if we add i_{r+1}^* , but not if we add h_{r+1}^* .
- If A_r has an odd number of i^* 's, we change the sign if we add h_{r+1}^* , but not if we add i_{r+1}^* .

In the following we denote for $S \in \mathfrak{S}_n$ respectively $T \in \mathfrak{T}_n$ the signed n -form by \tilde{S} respectively \tilde{T} . We observe, that if S and T only differ at position r , for the signs holds:

$$\text{sign}(\tilde{S}) = (-1)^{n-r+1} \cdot \text{sign}(\tilde{T}).$$

This is true, as after the step r of adding h^* or i^* , the two forms have different signs. In each of the following steps one of the forms will change its sign and the other one will not. So the signs of \tilde{S} and \tilde{T} differ if and only if the number $n - r$ of remaining steps is even.

Now we arrived at the point where we are able to define our candidate for the $H_{\mathbb{H}}^n$ -invariant $(n+1)$ -form η :

$$\eta := (-1)^n \cdot \left(\sum_{S \in \mathfrak{S}_n} K^* \wedge \tilde{S} - \sum_{T \in \mathfrak{T}_n} J^* \wedge \tilde{T} \right)$$

The only form in $\mathfrak{S}_n \cup \mathfrak{T}_n$, which is not zero when restricted to $H_{\mathbb{C}}^n$, is

$$h_1^* \wedge h_2^* \wedge \dots \wedge h_n^* \in \mathfrak{S}_n.$$

As the sign of this form is “+”, the form η coincides with γ on $H_{\mathbb{C}}^n$.

Further holds

$$s_t^* \eta = t^{n+2} \eta$$

as each summand in η consists of n dual forms of vectors of the first layer of the grading of the Lie algebra $\mathfrak{h}_{\mathbb{H}}^n$ which scale linearly and one dual form of a vector of the second layer of the grading which scales quadratically.

It remains to show that η is closed. As for this purpose the sign $(-1)^n$ has no effect, we will neglect it in the following.

For fixed $r \in \{1, \dots, n\}$ let $(S_p)_{p \in P}$ be a numbering of the forms $S \in \mathfrak{S}_n$ containing h_r^* . Then for each $p \in P$ there is a unique $T_p \in \mathfrak{T}_n$ containing i_r^* , such that S_p and T_p only differ at position r . This means T_p arises from S_p by just replacing h_r^* by i_r^* . This gives a numbering $(T_p)_{p \in P}$ of the forms $T \in \mathfrak{T}_n$ containing i_r^* associated to the numbering $(S_p)_{p \in P}$. Analogously let $(S_q)_{q \in Q}$ be a numbering of the $S \in \mathfrak{S}_n$ containing i_r^* . Then for each $q \in Q$ there is a unique $T_q \in \mathfrak{T}_n$ containing h_r^* , such that S_q and T_q only differs at position r . This means that T_q arises from S_q by just replacing i_r^* by h_r^* . This gives a numbering $(T_q)_{q \in Q}$ of the forms $T \in \mathfrak{T}_n$ containing h_r^* associated to the numbering $(S_q)_{q \in Q}$.

Further let \mathbf{p} and \mathbf{q} be the $(n-1)$ -forms obtained by deleting the position in which S_p and T_p differ and the position in which S_q and T_q differ, respectively. (Here it makes no difference whether one does this in the respective form $T \in \mathfrak{T}_n$ or $S \in \mathfrak{S}_n$.)

Then holds:

$$\begin{aligned} (n+2)! d\eta &= \sum_{S \in \mathfrak{S}_n} \sum_{t=1}^n (k_t^* \wedge h_t^* + j_t^* \wedge i_t^*) \wedge \tilde{S} - \sum_{T \in \mathfrak{T}_n} \sum_{t=1}^n (j_t^* \wedge h_t^* + i_t^* \wedge k_t^*) \wedge \tilde{T} \\ &= \sum_{t=1}^n \left(\sum_{S \in \mathfrak{S}_n} (k_t^* \wedge h_t^* + j_t^* \wedge i_t^*) \wedge \tilde{S} - \sum_{T \in \mathfrak{T}_n} (j_t^* \wedge h_t^* + i_t^* \wedge k_t^*) \wedge \tilde{T} \right) \\ &= \sum_{t=1}^n \left(\sum_{\substack{S \in \mathfrak{S}_n \\ \text{with } h_t^*}} j_t^* \wedge i_t^* \wedge \tilde{S} + \sum_{\substack{S \in \mathfrak{S}_n \\ \text{with } i_t^*}} k_t^* \wedge h_t^* \wedge \tilde{S} \right. \\ &\quad \left. - \sum_{\substack{T \in \mathfrak{T}_n \\ \text{with } h_t^*}} i_t^* \wedge k_t^* \wedge \tilde{T} - \sum_{\substack{T \in \mathfrak{T}_n \\ \text{with } i_t^*}} j_t^* \wedge h_t^* \wedge \tilde{T} \right) \\ &= \sum_{t=1}^n \left(\sum_{p \in P} [j_t^* \wedge i_t^* \wedge \tilde{S}_p - j_t^* \wedge h_t^* \wedge \tilde{T}_p] \right. \\ &\quad \left. + \sum_{q \in Q} [k_t^* \wedge h_t^* \wedge \tilde{S}_q - i_t^* \wedge k_t^* \wedge \tilde{T}_q] \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{t=1}^n \left(\sum_{p \in P} \left[(-1)^{x_p} \cdot j_t^* \wedge i_t^* \wedge h_t^* \wedge \mathfrak{p} - (-1)^{y_p} \cdot j_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{p} \right] \right. \\
&\quad \left. + \sum_{q \in Q} \left[(-1)^{x_q} \cdot k_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{q} - (-1)^{y_q} \cdot i_t^* \wedge k_t^* \wedge h_t^* \wedge \mathfrak{q} \right] \right)
\end{aligned}$$

where for each $t \in \{1, \dots, n\}$ the family $(S_p)_{p \in P}$ is a numbering of the $S \in \mathfrak{S}_n$ containing h_t^* and the family $(S_q)_{q \in Q}$ is a numbering of the $S \in \mathfrak{S}_n$ containing i_t^* .

Before we continue the computation, we have to identify the signs of the summands. We do this by computing the congruence classes modulo 2 of exponents of the -1 's. These exponents arise by pulling the h_t^* 's and i_t^* 's from inside of S_p, S_q, T_p and T_q to the third positions.

Denote by $\#_h^m A$ the number of h^* 's in $A \in \mathfrak{S}_n \cup \mathfrak{T}_n$ with index smaller than m and by $\#_i^m A$ the number of i^* 's in $A \in \mathfrak{S}_n \cup \mathfrak{T}_n$ with index smaller than m . Then for fixed t we have for the exponents x_p, x_q, y_p and y_q the following congruences modulo 2:

$$x_p = \#_h^t S_p$$

$$\begin{aligned}
y_p &= \text{sign}(\tilde{S}_p) - \text{sign}(\tilde{T}_p) + \#_h^n T_p + \#_i^t T_p \\
&\equiv (n - t + 1) + (n + 1) + (t - 1 - \#_h^t S_p) \\
&\equiv 2n + 1 - \#_h^t S_p \\
&\equiv 1 + \#_h^t S_p
\end{aligned}$$

$$\begin{aligned}
x_q &= \text{sign}(\tilde{S}_q) - \text{sign}(\tilde{T}_q) + \#_h^n S_q + \#_i^t S_q \\
&\equiv (n - t + 1) + (n) + (t - 1 - \#_h^t T_q) \\
&\equiv 2n - \#_h^t T_q \\
&\equiv \#_h^t T_q
\end{aligned}$$

$$y_q = \#_h^t T_q$$

We used for this, that S_p and T_p respectively S_q and T_q only differ at position t and so one has $\#_h^t S_p = \#_h^t T_p$ and $\#_h^t S_q = \#_h^t T_q$.

Further holds for the needed permutations the following:

$$\begin{aligned}
j_t^* \wedge i_t^* \wedge h_t^* &= (-1)^1 \cdot j_t^* \wedge h_t^* \wedge i_t^* \\
k_t^* \wedge h_t^* \wedge i_t^* &= (-1)^2 \cdot i_t^* \wedge k_t^* \wedge h_t^*
\end{aligned}$$

Therefore we can continue with the computation:

$$\begin{aligned}
(n+2)! d\eta &= \sum_{t=1}^n \left(\sum_{p \in P} \left[(-1)^{x_p} \cdot j_t^* \wedge i_t^* \wedge h_t^* \wedge \mathfrak{p} - (-1)^{y_p} \cdot j_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{p} \right] \right. \\
&\quad \left. + \sum_{q \in Q} \left[(-1)^{x_q} \cdot k_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{q} - (-1)^{y_q} \cdot i_t^* \wedge k_t^* \wedge h_t^* \wedge \mathfrak{q} \right] \right) \\
&= \sum_{t=1}^n \left(\sum_{p \in P} (-1)^{\#_h S_p} \cdot \left[j_t^* \wedge i_t^* \wedge h_t^* \wedge \mathfrak{p} - (-1)^1 \cdot j_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{p} \right] \right. \\
&\quad \left. + \sum_{q \in Q} (-1)^{\#_h T_q} \cdot \left[k_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{q} - i_t^* \wedge k_t^* \wedge h_t^* \wedge \mathfrak{q} \right] \right) \\
&= \sum_{t=1}^n \left(\sum_{p \in P} (-1)^{\#_h S_p} \cdot \left[j_t^* \wedge i_t^* \wedge h_t^* \wedge \mathfrak{p} - (-1)^{1+1} \cdot j_t^* \wedge i_t^* \wedge h_t^* \wedge \mathfrak{p} \right] \right. \\
&\quad \left. + \sum_{q \in Q} (-1)^{\#_h T_q} \cdot \left[k_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{q} - (-1)^2 \cdot k_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{q} \right] \right) \\
&= 0
\end{aligned}$$

So η is closed and therefore the conditions of Theorem 2.3 are fulfilled and we get $F_{H_{\mathbb{H}}^n}^{n+1}(l) \succ l^{\frac{n+2}{n}}$. \square

Together with the upper bound $F_{H_{\mathbb{H}}^n}^{n+1}(l) \preccurlyeq l^{\frac{n+2}{n}}$ from [3] this proves Theorem 1.

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